SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2009
YEAR 12 Mathematics Extension 1
HSC Task #2

Mathematics Extension 1

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Each Section is to be returned in a separate bundle.
- All answers must be given in exact simplified form unless otherwise stated.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All necessary working should be shown in every question.

Total Marks – 61

- Attempt questions 1-3
- Start each new section of a separate answer booklet

Examiner: *D.McQuillan*

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a} \sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a} \cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a} \tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a} \sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

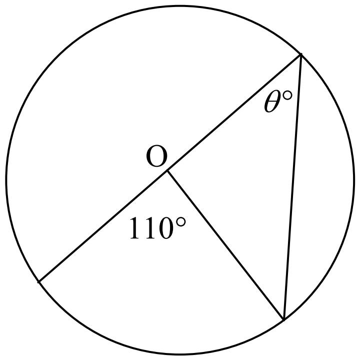
$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

Section A (21 marks)

Marks

(1) O is the centre of the circle. Find the value of θ. 1



(2) Find 5

(a) $\int 3x^5\,dx$

(b) $\int \frac{(3x+2)^2}{3}\,dx$

(c) $\int \frac{dx}{\sqrt{7x-1}}$

(3) Evaluate 5

(a) $\int_{-8}^{8} \frac{x^3}{2}\,dx$

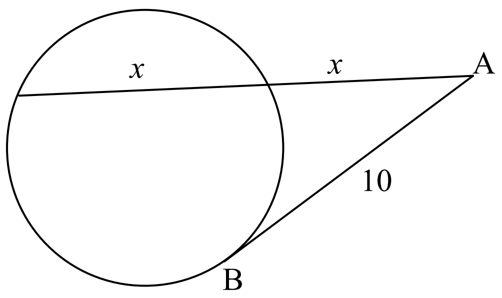
(b) $\int_{-3}^{3} \left(x^2+3\right)dx$

(c) $\int_0^4 \sqrt{4-x}\,dx$

(4) The new NSW number plates have three letters then two numbers followed by a letter. 1

How many number plates with this configuration are possible?

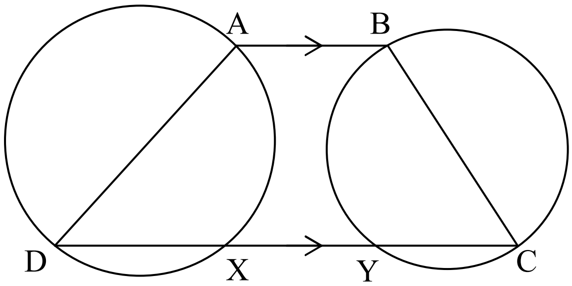
(5) AB is a tangent. Find the value of x. 2



(6) Given the parametric equations $x = 2t - 1$ and $y = 4t^2$ find $\frac{dy}{dx}$. 2

(7) Find the volume when $y = x^2 - 3$ is rotated about the y-axis between $y = -3$ and $y = 2$. 2

(8) ABCD is a trapezium in which AB || DC. AD and BC are the diameters of their respective circles, and these circles cut DC at X and Y respectively. Prove that ABYX is a rectangle. 3



End of Section A

Start a new ANSWER BOOKLET

Section B (20 marks)

Marks

(1) Use Simpson's Rule with 3 function values to find an approximation to the area under the arc of $y = \log_{10} x$ between $x = 2$ and $x = 4$. Round answer to 2 decimal places. 2

(2) For the function $f(x) = x^3 - 3x + 1$. 5

(a) Show that $f(x)$ has a zero for x between 0 and 1.

(b) Take $x = 0$ as your first approximation and use two applications of Newton's method to find a better approximation to 3 decimal places.

(c) Explain why you could not use $x = 1$ as your first approximation.

(3) A five-card hand is dealt from a standard 52-card deck. How many ways can you be dealt, 3

(a) three hearts and two clubs.

(b) three of one suit and two of another.

(4) Find the area enclosed between $y = x + 2$ and $y = x^2 - 2x + 2$. 3

(5) If ${}^nP_k = 120\,{}^nC_k$, find the value of k. 2

(6) Given that the equation for the chord of contact is $xx_0 = 2a\left(y + y_0\right)$.
Find the point of intersection of the tangents to $x^2 = 8y$ at the points where the chord of contact $y = x - 1$ intersects the parabola. 2

(7) For the parabola $x^2 = 4ay$. 3

(a) Derive the equation of the chord joining the points $P = \left(2ap, ap^2\right)$ and $Q = \left(2aq, aq^2\right)$.

(b) Show that as q approaches p the equation of the chord becomes the equation of the tangent.

End of Section B

Start a new ANSWER BOOKLET

Section C (20 marks)

Marks

(1) Prove that ${}^{n}P_k = {}^{n-1}P_k + k \cdot {}^{n-1}P_{k-1}$. 2

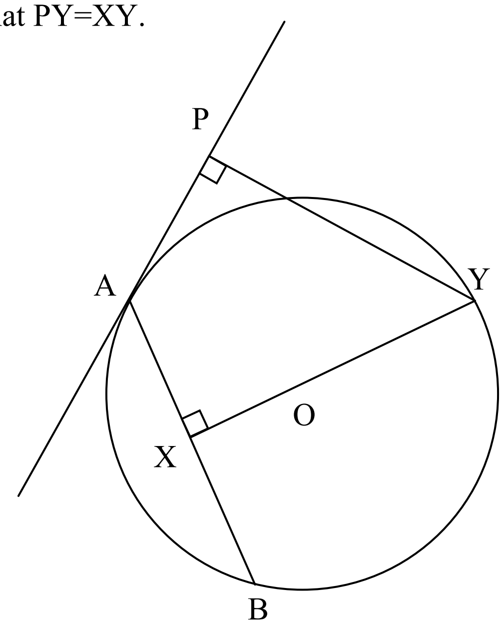
(2) AB is a chord of a circle and PA is a tangent at A. XY is a perpendicular bisector of AB and passes through the centre of the circle, O. PY is perpendicular to PA. 6

(a) Copy the diagram into your answer booklet.

(b) Show that triangle AYB is an isosceles triangle.

(c) If $\angle PAY = \alpha$ show that $\angle YAB = \alpha$.

(d) Prove that PY=XY.



(3) In how many ways can the letters of the word TOMATO be arranged if the Ts are to be separated? 2

(4) A Rugby League team of 13 players are to be selected at random from 20 boys. What is the probability that of 3 friends, 2 are selected and one misses out. 2

(5) For the parabola $x = 4t$, $y = 2t^2$. 4

(a) Derive the equations of the normals at $t = p$ and $t = -p$.

(b) The normals intersect at R. Find the locus of R.

(6) Use mathematical induction to prove that 4

$$\frac{1}{2} + \frac{2}{3} + \cdots + \frac{n}{n+1} < \frac{n^2}{n+1}$$

for all integers $n \geq 2$.

End of Section C

End of Exam

SECTION A

1/ $\theta = 55^\circ$ 1

2/ (a) $\int 3x^5\,dx = \frac{x^6}{2} + c$ 1

(b) $\int \frac{(3x+2)^2}{3}\,dx = \frac{(3x+2)^3}{27} + c$ 2

or $= x^3 + 2x^2 + \frac{4}{3}x + c$

(c) $\int \frac{dx}{(7x-1)^{\frac{1}{2}}} = \frac{2}{7}\sqrt{7x-1} + c$ 2

3/ (a) $\int_{-8}^{8} \frac{x^3}{2}\,dx = \left[\frac{x^4}{8}\right]_{-8}^{8} = 0$ 2

[Could use $\int_{-a}^{a} f(x)\,dx = 0$ if $f(x)$ odd]

(b) $\int_{-3}^{3} (x^2+3)\,dx = \left[\frac{x^3}{3} + 3x\right]_{-3}^{3}$ 2

$= \left[(9+9) - (-9-9)\right]$

$= 36$

[Could use $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(x)$ even]

(c) $\int \sqrt{x-4}\,dx = \frac{2}{3}(x-4)^{3/2} + c$ 1

4/ $26.26.26.10.10.26 = 45697600$ 1

5/ $(AB)^2 = 2x \times x$ 1

$100 = 2x^2$

$x = \sqrt{50} = 5\sqrt{2}$ 1

6/ $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

$= 8t \div 2$ 2

$= 4t$

$= 4\left(\frac{x+1}{2}\right)$

$= 2x + 2$

7/ $V = \pi \int_{-3}^{2} (y+3)\,dy$

$= \pi \left[\frac{y^2}{2} + 3y\right]_{-3}^{2}$ 1

$= \pi\left[(2+6) - (4.5 - 9)\right]$

$= 12.5\pi$ units^3 1

8/ 3

$A\hat{X}D = 90^\circ$ (angle in semi-c)

$A\hat{X}Y = 90^\circ$ (adj. angles on str. line)

Similarly

$B\hat{Y}C = 90^\circ$ (angle in semi-c)

$\therefore B\hat{Y}X = 90^\circ$ (adj. ang. on str. line)

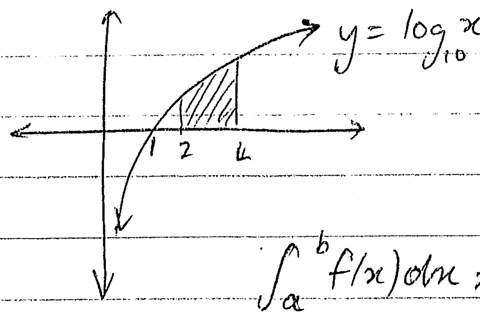
$\Rightarrow AX \parallel BY$ since the cointerior angles AXY and BYX are supplementary.

$\therefore$ ABYX is a //gram with one angle a right angle.

ie a rectangle.

Section B

1)



$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\tfrac{a+b}{2}\right) + f(b)\right]$$

x	2	3	4
$f(x)$	$\log_{10} 2$	$\log_{10} 3$	$\log_{10} 4$

$$\int_2^4 \log_{10} x\,dx \approx \frac{4-2}{6}\left[\log_{10} 2 + 4\log_{10} 3 + \log_{10} 4\right]$$

$$\approx 0.94 \text{ units}^2$$

2) a) $f(x) = x^3 - 3x + 1$

$f(0) = 1$

$f(1) = -1$

since $f(x)$ is continuous and opposite in sign at endpoints. $f(x)$ has at least one zero when $0 \leq x \leq 1$.

b) $a_1 = a - \dfrac{f(a)}{f'(a)}$

$f(x) = x^3 - 3x + 1$

$f(0) = 1$

$f'(x) = 3x^2 - 3$

$f'(0) = -3$

$a_1 = 0 - \dfrac{1}{-3}$

$a_1 = \dfrac{1}{3}$

$f\left(\tfrac{1}{3}\right) = \dfrac{1}{27}$

$f'\left(\tfrac{1}{3}\right) = -\dfrac{8}{3}$

$a_2 = \dfrac{1}{3} - \dfrac{\left(\frac{1}{27}\right)}{\left(-\frac{8}{3}\right)}$

$a_2 = \dfrac{25}{72}$

$a_2 \approx 0.347$

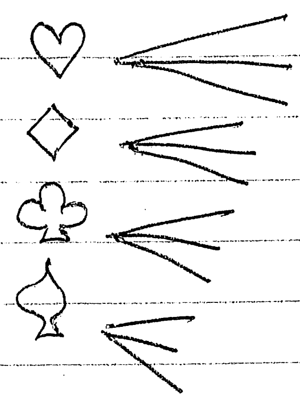
c) $f'(1) = 0$ tangent when $x=1$ is horizontal

$\therefore$ doesn't cross the x-axis.

$\therefore$ Newton's method fails.

3) a) ${}^{13}C_3 \times {}^{13}C_2 = 22308$

b) 3 cards 2 cards

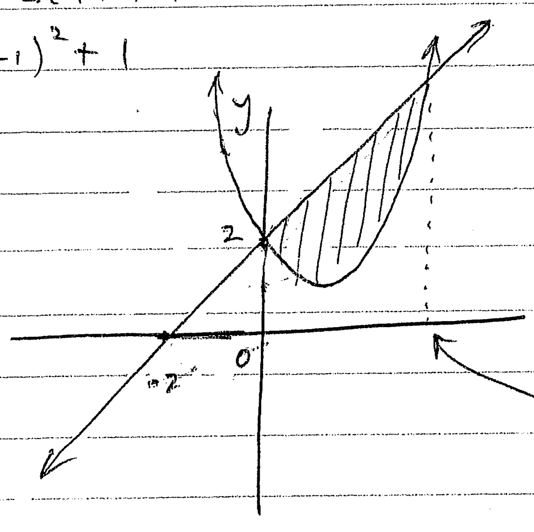


$12 \times 22308 = 267696$

4) $y = x^2 - 2x + 2$ —— ① $y = x + 2$ —— ②

$y = x^2 - 2x + 1 + 1$

$y = (x-1)^2 + 1$



sub ① into ②

$x^2 - 2x + \cancel{2} = x + \cancel{2}$

$x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0$, $x = 3$

$$\text{Area} = \int_0^3 \left[x + 2 - (x^2 - 2x + 2)\right] dx$$

$$= \int_0^3 (3x - x^2)\, dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3$$

$$= \frac{3(3)^2}{2} - \frac{(3)^3}{3} - (0)$$

$$= \frac{9}{2} \text{ units}^2$$

5) $^nP_k = 120\,{}^nC_k$

$$\cancel{\frac{n!}{(n-k)!}} = 120\,\cancel{\frac{n!}{(n-k)!}}\,\frac{1}{k!}$$

$$k! = 120$$

$$\therefore k = 5$$

6) $x^2 = 8y$ $\qquad$ $xx_1 = 2a(y+y_1)$

$x^2 = 4ay$ $\qquad$ $xx_1 = 4(y+y_1)$ $\qquad$ $y = x-1$

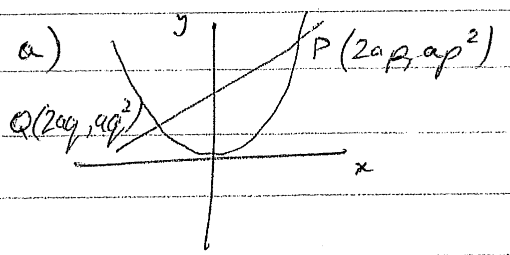
$\underline{\underline{a=2}}$ $\qquad$ equate $\qquad$ $x = y+1$

$$4x = 4(y+1)$$

$$x_1 = 4,\ y_1 = 1$$

$\therefore$ point of intersection of the tangents is $(4,1)$

7) a)



$$m_{PQ} = \frac{ap^2 - aq^2}{2ap - 2aq}$$

$$= \frac{a(p^2-q^2)}{2a(p-q)}$$

$$= \frac{(p-q)(p+q)}{2(p-q)}$$

$$= \frac{p+q}{2}$$

$$y - y_1 = m(x - x_1)$$

$$y - ap^2 = \frac{p+q}{2}(x - 2ap)$$

$y - \cancel{ap^2} = \frac{1}{2}(p+q)x - \cancel{ap^2} - apq$

$y - \frac{1}{2}(p+q)x + apq = 0$

b) as $q \to p$

$y - \frac{1}{2}(p+(p))x + ap(p) = 0$

$y - px + ap^2 = 0$

which is the equation of the tangent.

SECTION C. (EXT. 1.)

Q1

$$\underline{RHS} = {}^{n-1}P_k + k\,{}^{n-1}P_{k-1}$$

$$= \frac{(n-1)!}{(n-1-k)!} + k\frac{(n-1)!}{(n-k)!}$$

$$= \frac{(n-1)!}{(n-k)!}\left[n-k+k\right]$$

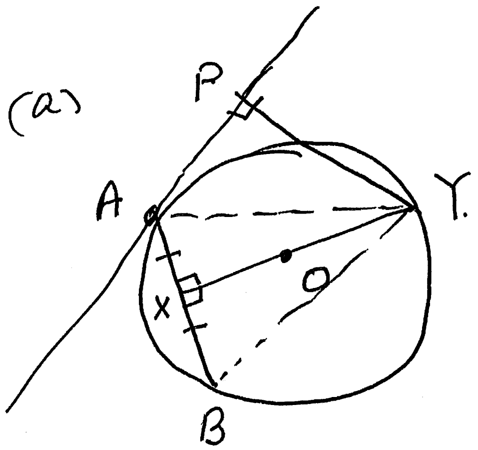
$$= \frac{n(n-1)!}{(n-k)!}$$

$$= \frac{n!}{(n-k)!}$$

$$= {}^nP_r$$

$$= \underline{LHS.}$$

Q2. (a)



(b) $AX = BX$ (data)

$\hat{AXY} = \hat{BXY}$ (data)

XY is common.

$\therefore \triangle AXY \equiv \triangle BXY$ (S.A.S)

$\therefore AY = BY$ (corresponding sides of congruent triangles)

$\therefore \triangle AYB$ is isosceles.

(c) $\angle PAY = \alpha$.

$\therefore \angle ABY = \alpha$ (angle between tangent and chord is equal to the angle in the alternate segment)

now $\angle YAB = \angle ABY$ (~~isosceles triangle~~, base angles equal).

$\therefore \angle YAB = \alpha$.

(d) $APYX$ is a cyclic quadrilateral (opposite angles $A\hat{P}Y$ and $A\hat{X}Y$ are supplementary)

$\therefore$ $XY = PY$ (equal angles are subtended by equal chords, at the circumference of a circle).

[OR. Show that $\triangle AXY \equiv \triangle APY$ using (AAS) test]

Q3. $\dfrac{6!}{2! \times 2!} - \dfrac{5!}{2!} = 120$. OR. $\dfrac{4!}{2!} \times \binom{5}{2} = 120$.

Q4. $\dfrac{\binom{3}{2} \times \binom{17}{11}}{\binom{20}{13}} = \dfrac{91}{190}$.

Q5. Given $y = 2t^2$ & $x = 4t$.

$\dfrac{dy}{dt} = 4t$. $\dfrac{dx}{dt} = 4$ $\dfrac{dt}{dx} = \dfrac{1}{4}$

$\therefore \dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx}$
$= 4t \times \frac{1}{4}$
$= t$.

$\therefore$ slope of normal where $t = p$ is $-\dfrac{1}{p}$.

$\therefore$ eqn. of normal is $\dfrac{y - 2p^2}{x - 4p} = -\dfrac{1}{p}$.

$py - 2p^3 = -x + 4p$.

$x + py = 2p^3 + 4p$ —(1).

at $t = -p$. $\dfrac{y - 2p^2}{x + 4p} = \dfrac{1}{p}$ $\therefore$ $py - 2p^3 = x + 4p$

$-x + py = 2p^3 + 4p$ —(2)

add ① & ②

$$2py = 4p^3 + 8p$$

$$y = 2p^2 + 4$$

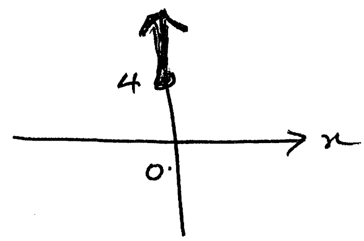
Subtract ② from ①.

$$2x = 0.$$

$$x = 0$$

$\therefore$ R is $(0, 2p^2+4)$

now the locus of R is the y-axis, where $y \geq 4$.



Q6. Step I. Consider $n = 2$.

$$LHS = \frac{1}{2} + \frac{2}{3} = \frac{7}{6} \qquad RHS = \frac{4}{3}$$

$\therefore$ true when $n = 2$ as $\frac{7}{6} < \frac{4}{3}$

Step II Assume the statement to be true when $n = k$.

ie. $\frac{1}{2} + \frac{2}{3} + \cdots + \frac{k}{k+1} < \frac{k^2}{k+1}$.

Step III Required to prove the statement is true for $n = k+1$ under the assumption in Step II.

ie. $\frac{1}{2} + \frac{2}{3} + \cdots + \frac{k}{k+1} + \frac{k+1}{k+2} < \frac{(k+1)^2}{k+2}$.

now

$$LHS = \frac{1}{2} + \frac{2}{3} + \cdots + \frac{k}{k+1} + \frac{k+1}{k+2}$$

$$< \frac{k^2}{k+1} + \frac{k+1}{k+2} \quad \text{(from assumption)}$$

$$= \frac{k^2(k+2) + (k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k^3 + 2k^2 + k^2 + 2k + 1}{(k+1)(k+2)}$$

$$< \frac{k^3 + 3k^2 + 3k + 1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^3}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{k+2} = RHS.$$

$$\therefore \quad \frac{1}{2} + \frac{2}{3} + \cdots + \frac{k+1}{k+2} < \frac{(k+1)^2}{k+2}.$$

<u>Step IV.</u> By the Principle of mathematical Induction the statement is true for all integral $n \geq 2$.